Optimal N-Impulse Transfer Trajectories Using Kustaanheimo/Stiefel Variables

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The necessary equations in Kustaanheimo/Stiefel (K/S) variables (E-formulation) are developed for an optimal N-impulse solution of the time-open and rendezvous problems. The equations are particularly simple when only two impulses are assumed. The developed algorithm is extremely fast and reliable, because the equations that must be solved are smooth, well-behaved (almost sinusoidal) functions of the variables. Numerical results for a planar, two-impulse, time-open and rendezvous problem are presented.

Introduction

CONSIDER the following problem: What is the sequence of impulses (times, locations, magnitudes, and directions) that will transfer a particle from an initial orbit defined by a state vector, $R(t_0)$, $\dot{R}(t_0)$, to a final orbit defined by a state vector, $R(t_F)$, $\dot{R}(t_F)$? This problem already has been solved by a number of investigators, using a variety of analytical and numerical approaches. Probably the most general approach is using primer vector theory.[1] This approach is based on Lawden's primer vector[2] to define an optimal trajectory, and on Lion and Handelsman's method[3] for defining and improving a nonoptimal trajectory. The chief disadvantage of this technique is the sheer size of the computer program which is necessary to solve the problem. The size of the program is dictated not only by the basic equations describing the two-body motion (Newtonian, Cartesian), but also by the unduly complicated equations describing the auxiliary calculations (Lambert's problem, transition matrices, etc.) of the state and costate variables across a trajectory.

An alternate approach for solving optimal impulsive trajectories is to formulate the problem in terms of a transformed set of variables – specifically, the K/S variables. Because the differential equations describing two-body motion in K/S variables are linear and can be integrated readily and expressed in terms of constants (elements) and elementary functions, the auxiliary calculations are trivially simple. Thus, Kepler's problem and the generation of the transition matrix are trivial calculations.[4] The K/S equations can be simplified further if the generalized eccentric anomaly, E (the eccentric anomaly plus a constant) is chosen as the particular independent variable. With this independent variable, the Keplerian energy equation is separable; that is, the energy can be expressed as an explicit function of the independent variable E. Lambert's problem then is the solution of a single, well-behaved (but still nonlinear) equation in one unknown.[5] This solution, however, is restricted to nonpositive energies – a minor restriction, since we are seldom interested in minimum impulsive solutions that involve hyperbolic paths.

The optimal N-impulse transfer problem using K/S variables will be solved by the following technique: A performance function will be defined (sum of the applied velocity changes) which is a function of only the arc lengths E_i and the interior impulse position vectors of the conics of the N-impulse solution. The gradient of this performance function with respect to these variables defines the necessary equations for an extremum. These equations will be solved numerically for the time-open transfer and the rendezvous problem. For the particular planar two-impulse example illustrated, the time-open transfer between two orbits will be shown to involve the solution of only three equations in three unknowns, and the rendezvous problem will involve the solution of only two equations in two unknowns. (The arc lengths or the times are not all independent in a rendezvous problem.)

Numerical results are presented, not only to verify the results of this solution with the Optimal Multi-Impulse Rendezvous (OMIR) program,[1] but also to illustrate that the equations that must be solved are smooth, well-behaved functions of the independent parameters.

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Problem Formulation

Performance Index

The performance index for an optimal N-impulse solution is defined as

$$\phi = \sum_{i=1}^{N} \Delta V_i^T \Delta V_i \tag{1}$$

where ΔV is the vector velocity change or the impulse, the subscript i refers to the time (t) of the particular impulse (see Fig. 1), N refers to the total number of impulses, and the superscript T refers to the transpose. The time intervals $\tau_i = t_{i+1} - t_i$ are the optimal coast times on the conics; the particular intervals τ_0 and τ_F are the coast times in the initial and final orbits, respectively.

An impulse is defined as $\Delta V = \dot{R}^+ - \dot{R}^-$, where R is the rectangular Cartesian position vector, the dot notation refers to differentiation with respect to t, and the superscripts minus and plus indicate a quantity evaluated immediately before and after an event, respectively. A velocity vector $\dot{R}$, expressed in K/S variables with the generalized eccentric anomaly E as the independent variable, is[4]

$$\dot{R} = (4\omega/r)\mathcal{L}(U)V$$

where $2\omega^2$ is the negative Keplerian energy. That is

$$2\omega^2 = [\mu - 8\omega^2(V, V)]/r \tag{2}$$

$\mathcal{L}(U)$ is the K/S transformation matrix, and U and $V = U^*$ are the four-dimensional position and velocity vectors in K/S space. (The fourth components of the vectors R and $\dot{R}$ are given values of zero.) The superscript ( )* refers to differentiation with respect to the generalized eccentric anomaly

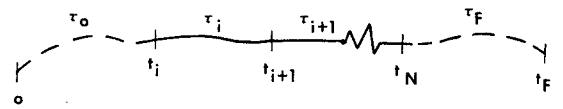

Fig. 1 Time sequence of impulses.

E. The symbol μ is the gravitational constant, and r is the magnitude of the position vector; that is

$$r = U^T U \equiv (U, U) \tag{3}$$

An impulse in K/S variables thus is expressed as

$$\Delta V = (4/r)\mathcal{L}(U)(\omega^+ V^+ - \omega^- V^-)$$

(Note that, by the definition of an impulse, quantities that are only position dependent may be factored from the expression; that is, $U^- = U^+ = U$.) The inner product of ΔV with itself is then

$$\Delta V^T \Delta V = (16/r^2)(\omega^+ V^+ - \omega^- V^-)^T$$

$$\mathcal{L}^T(U)\mathcal{L}(U)(\omega^+ V^+ - \omega^- V^-)$$

But

$$\mathcal{L}^T(U) = r\mathcal{L}^{-1}(U)$$

and the K/S transformation matrix may be eliminated from the square of the magnitude of the impulse

$$\Delta V^T \Delta V = (16/r)(\omega^+ V^+ - \omega^- V^-)^T(\omega^+ V^+ - \omega^- V^-)$$

It is convenient to define a K/S impulse vector I as

$$I = \omega^+ V^+ - \omega^- V^- \tag{4}$$

and a scalar ν as

$$\nu = I^T I / 2r \tag{5}$$

such that the performance index [Eq. (1)] may be expressed simply as

$$\phi = \sum_{i=1}^{N} \nu_i \tag{6}$$

where a convenient constant also has been introduced.

A few remarks on the K/S performance index are given in the following.

1) The K/S impulsive performance function with the generalized eccentric anomaly as the independent variable is not a simple function of the velocity changes in K/S variables.

2) The K/S impulses depend not only on the velocity vectors $V = U^*$, but also explicitly on the magnitude of the position vector r.

3) It should be emphasized that impulses occur in an infinitesimal time interval δt and not δE. The physical time t is a dependent variable and is related to the new independent variable E by the differential equation $t^* = r/2\omega$.

State Equations

In K/S variables with the generalized eccentric anomaly as the independent variable, the second-order differential equation that describes the motion of the state for the unperturbed two-body problem is

$$U^{**} + (U/4) = 0$$

The general solution to this equation is

$$U = \alpha\cos\theta + \beta\sin\theta \tag{7a}$$

$$V = (\beta\cos\theta - \alpha\sin\theta)/2 \tag{7b}$$

where the vectors α and β are constants (elements) and the independent variable $\theta = E/2$. For the conics indicated in Fig. 1 (those designated by the time intervals τ_0, τ_i $(i = 1,2,\ldots, N-1)$, and τ_F), the particular solutions for the state variables on the initial, transfer, and final conics are as follows:

Initial conic, $t_0 \le t \le t_1$:

$$U = \alpha_0\cos\theta_0 + \beta_0\sin\theta_0 \tag{8a}$$

$$V = (\beta_0\cos\theta_0 - \alpha_0\sin\theta_0)/2 \tag{8b}$$

Transfer conic, $t_i \le t \le t_{i+1}$ $(i = 1,2,\ldots,N-1)$:

$$U = \alpha_i\cos\theta_i + \beta_i\sin\theta_i \tag{9a}$$

$$V = (\beta_i\cos\theta_i - \alpha_i\sin\theta_i)/2 \tag{9b}$$

Final conic, $t_N \le t \le t_F$:

$$U = \alpha_F\cos\theta_F + \beta_F\sin\theta_F \tag{10a}$$

$$V = (\beta_F\cos\theta_F - \alpha_F\sin\theta_F)/2 \tag{10b}$$

where $\theta_0 = \theta(\tau_0)$, $\theta_i = \theta(\tau_i)$, and $\theta_F = \theta(\tau_F)$.

Boundary Conditions

On the initial conic, let the angle θ_0 be equal to zero at the time t_0. The elements α_0 and β_0 then can be calculated in terms of the boundary conditions on the initial orbit

$$\alpha_0 = U_0 \tag{11a}$$

$$\beta_0 = 2V_0 \tag{11b}$$

On the transfer conics, let the angles θ_i be equal to zero at the times t_i. The elements α_i and β_i can then be calculated in terms of the continuity conditions on the position vector U; that is,

$$U_i = \alpha_i$$

$$U_{i+1} = \alpha_i\cos\theta_i + \beta_i\sin\theta_i$$

Solving for α_i and β_i, we have†

$$\alpha_i = U_i \tag{12a}$$

$$\beta_i = (U_{i+1} - U_i\cos\theta_i)/\sin\theta_i \tag{12b}$$

(Note that $\theta_i \ne k\pi$ for $k = 0, 1, \ldots$)

On the final conic, let the angle θ_F be equal to zero at the time t_F. The elements α_F and β_F then can be calculated in terms of the boundary conditions on the final orbit

$$\alpha_F = U_F \tag{13a}$$

$$\beta_F = 2V_F \tag{13b}$$

Degrees of Freedom

A general condition will result if we compute the degrees of freedom M (difference between the total number of free parameters and the number of constraints) of a K/S, N-impulse problem. If the dimensions of the vectors α or β are n, then for a K/S, time-open, N-impulse solution between two orbits, the list of constraint equations and unknown parameters is as shown in Table 1. The number of degrees of

†The transfer conic elements, α_i and β_i, in R^3 space, are additionally constrained by a mapping condition in the form of a bilinear relation,[5] $\ell(U_i, U_{i+1}) = 0$ $(i = 1,2,\ldots,N-1)$.

Table 1 *K/S* constraint equations and unknown parameters for a time-open transfer

Type of condition	Constraint equation	Unknowns
Initial and final boundary conditions	$4n$; Eqs. (8), (10)	$4n$; α_0, β_0, α_F, β_F
Continuity conditions	$2n(N-1)$; Eq. (9)	$2n(N-1)$; α_i, β_i $(i=1,2,\ldots,N-1)$
Arc times	None	$(N-1)+2$; θ_i $(i=1,2,\ldots,N-1)$, θ_0, θ_F
Interior impulse position vectors	None	$n(N-2)$; U_{k+1} $(k=1,2,\ldots,N-2)$

freedom M is then $M=n(N-2)+(N+1)$. Thus, for $N>2$, the *K/S* solution appears to be at a disadvantage with respect to the Newtonian Cartesian solution, because it requires one additional free parameter for each additional interior impulse. This occurs, of course, because the *K/S* position vector U has four elements. However, the interior impulse position vectors have an additional constraint imposed upon them because of a mapping condition (see the footnote on the previous page). Hence, the number of degrees of freedom for the *K/S* solution is the same as for the Cartesian solution. Note that, for a two-impulse solution, the dimensions of the state space do not affect the degrees of freedom of the problem.

Problem Solution

Time-Open

By examining Eqs. (8-10), the elements of the performance function ν_i [Eq. (5)] can be noted to be a function of only the arc lengths of the three types of conics and the interior impulse position vectors. Expressed functionally, this is

$$\nu_i=\nu_i[\theta_0,\ \theta_F,\ \theta_j\ (j=1,2,\ \ldots,\ N-1),\ U_{k+1}\ (k=1,2,\ \ldots,\ N-2)]$$

The necessary condition for the performance index [Eq. (6)] to be an extremum is that the gradient of ϕ must be equal to zero. Differentiating the performance index with respect to these variables, and setting it equal to zero, we have

$$\mathrm{d}\phi=\sum_{i=1}^{N}\left[\frac{\partial\nu_i}{\partial\theta_0}\mathrm{d}\theta_0+\frac{\partial\nu_i}{\partial\theta_F}\mathrm{d}\theta_F+\sum_{j=1}^{N-1}\frac{\partial\nu_i}{\partial\theta_j}\mathrm{d}\theta_j+\sum_{k=1}^{N-2}\frac{\partial\nu_i}{\partial U_{k+1}}\mathrm{d}U_{k+1}\right]=0 \tag{14}$$

Since θ_0, θ_F, θ_j, and U_{k+1} all are independent parameters, this equation represents $[n(N-2)+(N-1)+2]$ necessary conditions for a time-open, optimal, N-impulse solution in *K/S* variables between two orbits. The satisfaction of these necessary conditions will produce a solution that 1) departs from the initial orbit at the optimal location, 2) has N impulses that are optimal with respect to time, location, magnitude, and direction, and 3) arrives on the final orbit at the optimal location.

Rendezvous

A rendezvous problem is a time-open problem with the additional constraint that the total time of transfer is determined or fixed. Expressed mathematically, this constraint is (see Fig. 1)

$$\tau_0+\sum_{i=1}^{N-1}\tau_i-\tau_F=t_F-t_0$$

where the time increment t_F-t_0 is a specified quantity.‡ The differential of this expression is

$$\mathrm{d}\tau_0+\sum_{i=1}^{N-1}\mathrm{d}\tau_i-\mathrm{d}\tau_F=0 \tag{15}$$

But from the general differential equation for the physical time in *K/S* variables, we have

$$2\omega\ \mathrm{d}\tau=r\ \mathrm{d}E=2r\ \mathrm{d}\theta$$

On a given conic, the elements α and β and the energy variable ω are constants; hence, the differential equation may be integrated readily. The result is[5]

$$2\omega\tau=b_1(1-\cos E)+b_2(E+m2\pi)+b_3\sin E \tag{16}$$

where the integer m refers to a general multiple revolution counter, and the b's are

$$b_1=(\alpha,\ \beta) \tag{17a}$$

$$b_2=1/2[(\alpha,\ \alpha)+(\beta,\ \beta)] \tag{17b}$$

$$b_3=1/2[(\alpha,\ \alpha)-(\beta,\ \beta)] \tag{17c}$$

Equation (16) relates the physical time interval τ to the new independent variable E. On the initial and final conics, the elements α_0, β_0, α_F, and β_F and the energy variables ω_0 and ω_F are specified constants; and hence the differential relationship between the τ and θ for the two conics is simply

$$\mathrm{d}\tau_0=(r_1/\omega_0)\mathrm{d}\theta_0 \tag{18a}$$

$$\mathrm{d}\tau_F=(r_N/\omega_F)\mathrm{d}\theta_F \tag{18b}$$

On the transfer conics, however, the elements α and β and the energy variables ω are functions of θ_0, θ_F, and θ_j $(j=1,2,\ldots,N-1)$ and the interior impulse position vectors U_{k+1} $(k=1,2,\ldots,N-2)$; and hence the differential relationship betweem the physical times τ_i and these variables may be expressed as

$$\mathrm{d}\tau_i=A_i\ \mathrm{d}\theta_0+B_i\ \mathrm{d}\theta_F+\sum_{j=1}^{N-1}C_{ij}\mathrm{d}\theta_j+\sum_{k=1}^{N-2}D_{i,k+1}\mathrm{d}U_{k+1}\qquad (i=1,2,\ldots,\ N-1) \tag{19}$$

where the A's, B's, C's, and D's are appropriate partial derivatives of Eq. (16).

By using the relationships for $\mathrm{d}\tau_0$, $\mathrm{d}\tau_F$, and $\mathrm{d}\tau_i$ in Eq. (15), the relationship between the parameters for a rendezvous problem is

$$\sum_{i=1}^{N-1}\left[\left(\frac{r_1}{\omega_0}+A_i\right)\mathrm{d}\theta_0+\sum_{j=1}^{N-1}C_{ij}\mathrm{d}\theta_j+\sum_{k=1}^{N-2}D_{i,k+1}\mathrm{d}U_{k+1}+\left(B_i-\frac{r_N}{\omega_F}\right)\mathrm{d}\theta_F\right]=0 \tag{20}$$

‡The sign of τ_F is negative because of the manner of choosing the final boundary conditions.

Using this equation to eliminate one of the differentials of the parameters (θ_0, θ_F, θ_j, U_{k+1}) will produce a solution that 1) departs from the initial orbit at an optimal location, 2) has N impulses that are optimal with respect to the time, location, magnitude, and direction, and 3) optimally rendezvous with the second vehicle in the final orbit.

Two-Impulse Transfers

Time-Open Solution

For a two-impulse solution, the performance function is [see Eq. (6)] simply

$$\phi = \nu_1 + \nu_2 = (I_1^T I_1 / 2r_1) + (I_2^T I_2 / 2r_2) \tag{21}$$

For $N=2$, there is only one intermediate or transfer conic; and hence there are no interior impulse position vectors in the control vector [see Eq. (14)]. For simplicity of nomenclature, the subscript on the arc length θ_i will be dropped; that is, $\theta_i = \theta_1 = \theta$. The independent parameters in the two-impulse problem then reduce to the generalized eccentric anomalies on the three conics – that is, θ_0, θ_F, and θ. From Eq. (14), the necessary equations (the gradient components of the performance function with respect to the independent parameters θ_0, θ, and θ_F) to be solved for the solution of a K/S, time-open, optimal, two-impulse problem between two orbits are defined as the vector $\boldsymbol{F}$ with elements

$$F_1 = \left(I_1^T \frac{\partial I_1}{\partial \theta_0} - \nu_1 \frac{\partial r_1}{\partial \theta_0} \right) \Big/ r_1 + \frac{I_2^T}{r_2} \frac{\partial I_2}{\partial \theta_0} = 0 \tag{22a}$$

$$F_2 = \frac{I_1^T}{r_1} \frac{\partial I_1}{\partial \theta} + \frac{I_2^T}{r_2} \frac{\partial I_2}{\partial \theta} = 0 \tag{22b}$$

$$F_3 = \frac{I_1^T}{r_1} \frac{\partial I_1}{\partial \theta_F} + \left(I_2^T \frac{\partial I_2}{\partial \theta_F} - \nu_2 \frac{\partial r_2}{\partial \theta_F} \right) \Big/ r_2 = 0 \tag{22c}$$

where the partial derivatives are developed in the appendix.

Rendezvous

For the two-impulse rendezvous problem between two orbits, the differentials of θ_0, θ, and θ_F are not all independent, but are related by the expression [see Eq. (20)]

$$\mathrm{d}\theta = k_1 \, \mathrm{d}\theta_0 + k_2 \, \mathrm{d}\theta_F \tag{23}$$

where

$$k_1 = -(r_1/\omega_0 + A)/C \tag{24a}$$

$$k_2 = -(B - r_2/\omega_F)/C \tag{24b}$$

$$A = [a_5(\alpha, V_1^-) + a_6(\beta, V_1^-) - \tau(\partial\omega/\partial\theta_0)]/\omega \tag{24c}$$

$$B = \{[a_1(\alpha, V_2^+) + a_4(\beta, V_2^+)]\csc\theta - \tau(\partial\omega/\partial\theta_F\}/\omega \tag{24d}$$

$$C = \{r_2 - [a_1(\alpha, V_2^-) + a_4(\beta, V_2^-)]\csc\theta - \tau(\partial\omega/\partial\theta)\}/\omega \tag{24e}$$

and

$$a_1 = 1 - \cos 2\theta \qquad a_2 = 2(\theta + m\pi) \qquad a_3 = \sin 2\theta$$

$$a_4 = a_2 - a_3 \qquad a_5 = a_2 + a_3 - a_1 \cot\theta \qquad a_6 = a_1 - a_4 \cot\theta$$

By using the vector $\boldsymbol{F}$, the gradient of the performance index may be expressed as

$$\mathrm{d}\phi = F_1 \mathrm{d}\theta_0 + F_2 \mathrm{d}\theta + F_3 \mathrm{d}\theta_F \tag{25}$$

If $\mathrm{d}\theta$ is eliminated from Eqs. (23) and (25) and the coefficients of $\mathrm{d}\theta_0$ and $\mathrm{d}\theta_F$ are set equal to zero, the necessary equations to be solved for a solution of the optimal two-impulse rendezvous problem in K/S variables between two orbits are given as

$$G_1 = F_1 + k_1 F_2 = 0 \tag{26a}$$

$$G_2 = F_3 + k_2 F_2 = 0 \tag{26b}$$

where F_1, F_2, and F_3 are given by Eq. (22).

Numerical Results

A planar transfer between two circular orbits with altitudes of 100 and 300 naut. miles, respectively, will be used to illustrate the K/S two-impulse time-open and rendezvous transfer problems. The initial position vector $\boldsymbol{R_0}$ is orientated with respect to the final position vector $\boldsymbol{R_F}$ by an angle of 119.9 deg.

Figure 2 is a plot of the performance function ϕ and its gradient $\nabla\phi = F_2(\theta)$, $(\theta_0 = \theta_F = 0)$ vs the generalized eccentric anomaly θ for this particular orientation of the two state vectors. Requiring θ_0 and θ_F to be equal to zero implies that the two impulses occur at the initial and final times, t_0 and t_F, respectively $(\tau_0 = \tau_F = 0)$ (see Fig. 1). Since the performance functions – ϕ and its gradient $\nabla\phi$ – are symmetric about $\theta = 0$, it is only necessary to examine the region $\theta = (0, \pi)$ to illustrate the solution.

Note the smoothness of these two functions, ϕ and $\nabla\phi$, in Fig. 2; they are almost sinusoidal functions of the variable θ. This characteristic also was noted in the solution of the K/S Lambert problem,[5] when the generalized eccentric anomaly was used as the independent variable. Three zeros are noted for the function $\nabla\phi$ or $F_2(\theta)$ in this region of θ. However, values of $\theta = k\pi$ $(k = 0, 1, 2, \ldots)$ already have been eliminated [see Eq. (12)] as singular points. They correspond to zero energy transfers (parabolic paths) and are local maximums of the performance function ϕ. Thus, there is only one valid zero for the function $F_2(\theta)$ that corresponds to the minimum value of ϕ. This value of ϕ (it appears to be zero in the figure) represents the minimum-cost $(\phi = 1057$ fps) time-open transfer.

It is now readily apparent why the K/S two-impulse algorithm is extremely fast and reliable. For if given any value of the independent variable, $\theta (0 < \theta < \pi)$, if a one-dimensional search is performed on the function ϕ, the solution always must proceed towards the single valid solution of $\boldsymbol{F}$ that represents the minimum-time two-impulse transfer soltuion.

In Fig. 3, an optimal K/S two-impulse rendezvous between two circular orbits is illustrated by the four curves A, B, C, and D.

Curve A is a plot of the performance function ϕ vs the rendezvous time, $t_F - t_0$, for a case in which the initial and

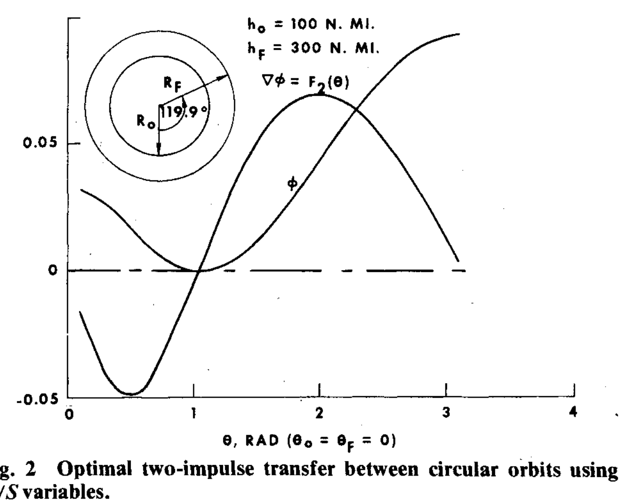

Fig. 2 Optimal two-impulse transfer between circular orbits using K/S variables.

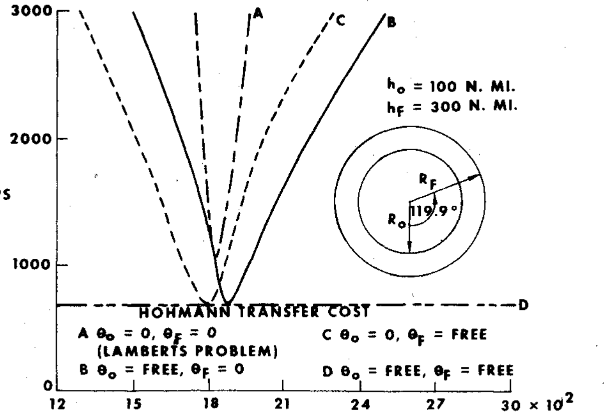

Fig. 3 Optimal two-impulse rendezvous between circular orbits using *K/S* variables.

final generalized eccentric anomalies, θ_0 and θ_F, respectively, are required to be zero. This, then, is a Lambert's problem between the two position vectors $\boldsymbol{R}_0$ and $\boldsymbol{R}_F$ for various rendezvous times $t_F - t_0$. For this region of the transfer time, there is a single minimum of the performance function ϕ ($\phi = 1057$ fps) at a rendezvous time of 1838 sec.

Curve B is similar to curve A; however, the initial generalized eccentric anomaly is allowed to be free. This solution has a minimum at the Hohmann rendezvous time ($t_F - t_0 = 1870$ sec), corresponding to a cost of 692.14 fps. When the initial generalized eccentric anomaly is allowed to be free, the optimum position vector $\boldsymbol{R}_1$ for the minimum cost is colinear with the final position vector $\boldsymbol{R}_F$; however, in the opposite direction. Thus, the minimum-cost solution is determined by a position vector $\boldsymbol{R}_1$, that is, π rad from $\boldsymbol{R}_F$ or a Hohmann transfer solution.

Curve C is similar to curve A; however, the final generalized eccentric anomaly is now allowed to be free. The minimum cost is the Hohmann cost resulting from a transfer from the initial position vector $\boldsymbol{R}_0$, to the second-impulse position vector $\boldsymbol{R}_2$ alined at an angle of the π rad with respect to $\boldsymbol{R}_0$.

Curve D is a combination of curves B and C; that is, the initial and final generalized eccentric anomalies are allowed to be free. The cost for this region of the rendezvous time $t_F - t_0$ (single revolution) is the Hohmann cost, since this transfer time always can be determined for the position vectors $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, which are separated by π rad.

What is amazing about Fig. 3 is not the results (they could have been obtained by a number of different methods), but the efficiency with which the data were obtained. The entire figure was generated in a fraction of the previously required time,[1] no data point requiring more than two or three derivative calculations.

Appendix: Derivation of the Necessary *K/S* Equations for a Time-Open, Optimal, Two-Impulse Transfer between Two Orbits

If given the three equations [Eqs. (22)]

$$F_1 = \partial\nu_1/\partial\theta_0 + \partial\nu_2/\partial\theta_0 = 0$$

$$F_2 = \partial\nu_1/\partial\theta + \partial\nu_2/\partial\theta = 0$$

$$F_3 = \partial\nu_1/\partial\theta_F + \partial\nu_2/\partial\theta_F = 0$$

where

$$\nu_1 = \boldsymbol{I}_1{}^T\boldsymbol{I}_1/2r_1 \qquad \nu_2 = \boldsymbol{I}_2{}^T\boldsymbol{I}_2/2r_2$$

and the impulse vectors are

$$\boldsymbol{I}_1 = \omega\boldsymbol{V}_1^+ - \omega_0\boldsymbol{V}_1^- \qquad \boldsymbol{I}_2 = \omega_F\boldsymbol{V}_2^+ - \omega\boldsymbol{V}_2^-$$

what are the expressions for F_1, F_2, and F_3? By using the definitions of ν_1 and ν_2, the vector $\boldsymbol{F}$ may be expressed as

$$F_1 = \left(\boldsymbol{I}_1{}^T\frac{\partial \boldsymbol{I}_1}{\partial\theta_0} - \nu_1\frac{\partial r_1}{\partial\theta_0}\right)\Big/ r_1 + \frac{\boldsymbol{I}_2{}^T}{r_2}\frac{\partial\boldsymbol{I}_2}{\partial\theta_0}$$

$$F_2 = \frac{\boldsymbol{I}_1}{r_1}\frac{\partial\boldsymbol{I}_1}{\partial\theta} + \frac{\boldsymbol{I}_2{}^T}{r_2}\frac{\partial\boldsymbol{I}_2}{\partial\theta}$$

$$F_3 = \frac{\boldsymbol{I}_1{}^T}{r_1}\frac{\partial\boldsymbol{I}_1}{\partial\theta_F} + \left(\boldsymbol{I}_2{}^T\frac{\partial\boldsymbol{I}_2}{\partial\theta_F} - \nu_2\frac{\partial r_2}{\partial\theta_F}\right)\Big/ r_2$$

where $r_1 = r_1(\theta_0)$ and $r_2 = r_2(\theta_F)$ [see Eqs. (8) and (10)].

The energy terms ω_0 and ω_F are constants that are computed from the boundary conditions on the initial and final orbits, respectively. The energy term ω on the transfer conic is defined as [see Eq. (2)]

$$2\omega^2 = \mu/[(\alpha,\ \alpha) + (\beta,\ \beta)]$$

The partial derivatives of the impulse vectors $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ and the radius magnitudes r_1 and r_2 with respect to θ_0, θ, and θ_F are

$$\frac{\partial\boldsymbol{I}_1}{\partial\theta_0} = \omega\frac{\partial\boldsymbol{V}_1^+}{\partial\theta_0} + \boldsymbol{V}_1^+\frac{\partial\omega}{\partial\theta_0} - \omega_0\frac{\partial\boldsymbol{V}_1^-}{\partial\theta_0}$$

$$\frac{\partial\boldsymbol{I}_1}{\partial\theta} = \omega\frac{\partial\boldsymbol{V}_1^+}{\partial\theta} + \boldsymbol{V}_1^+\frac{\partial\omega}{\partial\theta}, \qquad \frac{\partial\boldsymbol{I}_1}{\partial\theta_F} = \omega\frac{\partial\boldsymbol{V}_1^+}{\partial\theta_F} + \boldsymbol{V}_1^+\frac{\partial\omega}{\partial\theta_F}$$

$$\frac{\partial\boldsymbol{I}_2}{\partial\theta_0} = -\omega\frac{\partial\boldsymbol{V}_2^-}{\partial\theta_0} - \boldsymbol{V}_2^-\frac{\partial\omega}{\partial\theta_0}, \qquad \frac{\partial\boldsymbol{I}_2}{\partial\theta} = -\omega\frac{\partial\boldsymbol{V}_2^-}{\partial\theta} - \boldsymbol{V}_2^-\frac{\partial\omega}{\partial\theta}$$

$$\frac{\partial\boldsymbol{I}_2}{\partial\theta_F} = \omega_F\frac{\partial\boldsymbol{V}_2^+}{\partial\theta_F} - \omega\frac{\partial\boldsymbol{V}_2^-}{\partial\theta_F} - \boldsymbol{V}_2^-\frac{\partial\omega}{\partial\theta_F}$$

$$\frac{\partial r_1}{\partial\theta_0} = 2\left(\boldsymbol{U}_1,\ \frac{\partial\boldsymbol{U}_1}{\partial\theta_0}\right) = 4(\boldsymbol{U}_1,\ \boldsymbol{V}_1^-)$$

$$\frac{\partial r_2}{\partial\theta_F} = 2\left(\boldsymbol{U}_2,\ \frac{\partial\boldsymbol{U}_2}{\partial\theta_F}\right) = 4(\boldsymbol{U}_2,\ \boldsymbol{V}_2^+)$$

The partial derivatives of the velocity vectors $\boldsymbol{V}_1^+$, $\boldsymbol{V}_1^-$, $\boldsymbol{V}_2^+$, and $\boldsymbol{V}_2^-$ and the energy term ω with respect to θ_0, θ, and θ_F are [see Eqs. (2 and 8-10)]

$$\frac{\partial\boldsymbol{V}_1^-}{\partial\theta_0} = -\boldsymbol{U}_1/2, \quad \frac{\partial\boldsymbol{V}_1^+}{\partial\theta_0} = -\boldsymbol{V}_1^-\cot\theta, \quad \frac{\partial\boldsymbol{V}_1^+}{\partial\theta} = -\boldsymbol{V}_2^-\csc\theta$$

$$\frac{\partial\boldsymbol{V}_1^+}{\partial\theta_F} = \boldsymbol{V}_2^+\csc\theta, \quad \frac{\partial\boldsymbol{V}_2^-}{\partial\theta_0} = -\boldsymbol{V}_1^-\csc\theta, \quad \frac{\partial\boldsymbol{V}_2^-}{\partial\theta} = -\boldsymbol{V}_1^+\csc\theta$$

$$\frac{\partial\boldsymbol{V}_2^-}{\partial\theta_F} = \boldsymbol{V}_2^+\cot\theta, \quad \frac{\partial\boldsymbol{V}_2^+}{\partial\theta_F} = -\boldsymbol{U}_2/2, \quad \frac{\partial\omega}{\partial\theta_0} = \frac{2\omega}{b_2}(\boldsymbol{V}_1^-,\ \boldsymbol{V}_2^-)\csc\theta$$

$$\frac{\partial\omega}{\partial\theta} = \frac{2\omega}{b_2}(\boldsymbol{V}_1^+,\ \boldsymbol{V}_2^-)\csc\theta, \qquad \frac{\partial\omega}{\partial\theta_F} = -\frac{2\omega}{b_2}(\boldsymbol{V}_1^+,\ \boldsymbol{V}_2^+)\csc\theta$$

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